

COMMENT ON THE PHOTON NUMBER BOUND AND RAYLEIGH SCATTERING

JÉRÉMY FAUPIN AND ISRAEL MICHAEL SIGAL

ABSTRACT. We discuss photon number bounds for a system of non-relativistic particles coupled to the quantized electromagnetic field (non-relativistic QED), below the ionization threshold. Such a bound was assumed in the proof of asymptotic completeness for Rayleigh scattering in our paper [3] (Condition (1.20) of Theorem 1.3 in [3]). We show how this assumption can be weakened and verified for a class of hamiltonians.

1. INTRODUCTION

In this note we discuss photon number bounds for non-relativistic particle systems coupled to quantized electromagnetic or phonon field. (We use the term photon for both photon and phonon.) Such a bound was first proved by W. De Roeck and A. Kupiainen in [2] for the spin-boson model and a variant of such a bound was assumed in our proof of asymptotic completeness below the ionization threshold, i.e. for Rayleigh scattering, in [3]. Specifically, we assumed that the photon number is bounded uniformly in time (Condition (1.20) of Theorem 1.3 in [3]). In this note we show how this assumption can be weakened and verified for a class of hamiltonians.

In [3] we consider the dynamics generated by the Hamiltonian (here and in what follows we use, without mentioning it, the notation of the paper [3])

$$H = H_p + H_f + I(g), \quad (1.1)$$

acting on the state space $\mathcal{H} := \mathcal{H}_p \otimes \mathcal{F}$. Here, \mathcal{H}_p is the particle state space, \mathcal{F} is the bosonic Fock space based on the one-photon space $L^2(\mathbb{R}^3)$, H_p is a self-adjoint Hamiltonian acting on \mathcal{H}_p , and $H_f := d\Gamma(\omega)$ (where $\omega(k) = |k|$ is the photon dispersion law and k is the photon wave vector) is the photon Hamiltonian acting on \mathcal{F} .

The operator $I(g)$, acting on \mathcal{H} , represents an interaction energy labeled by a coupling family $g(k)$ of operators acting on \mathcal{H}_p . It is of the form

$$I(g) := \int (g^*(k) \otimes a(k) + g(k) \otimes a^*(k)) dk, \quad (1.2)$$

with $a^*(k)$ and $a(k)$ the creation and annihilation operators acting on \mathcal{F} . The coupling operators $g(k)$ are assumed to satisfy

$$\|\eta^{|\alpha|} \partial^\alpha g(k)\|_{\mathcal{H}_p} \lesssim |k|^{\mu-|\alpha|} \xi(k), \quad |\alpha| \leq 2, \quad (1.3)$$

where $\xi(k)$ is an ultraviolet cutoff (a smooth function decaying sufficiently rapidly at infinity) and η is an estimating operator (a bounded, positive operator with unbounded inverse) on \mathcal{H}_p , satisfying

$$\|\eta^{-n} f(H)\| \lesssim 1, \quad (1.4)$$

for any $n = 1, 2$ and $f \in C_0^\infty((-\infty, \Sigma))$, where Σ is the ionization threshold.

The proofs presented here, as well as - as was mentioned in [3] - those in [3], can be extended to the minimal coupling model with the standard quantum Hamiltonian (see [1] for the notations used)

$$H = \sum_{j=1}^n \frac{1}{2m_j} (-i\nabla_{x_j} - g_j A_\kappa(x_j))^2 + V(x) + H_f.$$

Let $\psi_t = e^{-itH}\psi_0$ be the solution of the Schrödinger equation $i\partial_t\psi_t = H\psi_t$ with an initial condition $\psi_0 \in \text{Ran } E_{(-\infty, \Sigma)}(H)$. The assumption (1.20) of Theorem 1.3 of [3] states that

- For any $\psi_0 \in D(N^{1/2})$ and uniformly in $t \in [0, \infty)$,

$$\|N^{1/2}\psi_t\| \lesssim \|N^{1/2}\psi_0\| + \|\psi_0\|. \quad (1.5)$$

It can be weakened to one of the following conditions:

- (i) (1.5) holds *only* for initial states $\psi_0 \in f(H)D(N^{1/2})$, with $f \in C_0^\infty((E_{\text{gs}}, \Sigma))$,
- (i') There exists a set \mathcal{D} such that $\mathcal{D} \cap D(d\Gamma(\omega^{-1/2}\langle y \rangle \omega^{-1/2})^{\frac{1}{2}})$ is dense in $\text{Ran } E_{(-\infty, \Sigma)}(H)$ and, for any $\psi_0 \in \mathcal{D}$,

$$\|d\Gamma(\omega^{-1})^{\frac{1}{2}}\psi_t\| \lesssim C(\psi_0), \quad (1.6)$$

uniformly in $t \in [0, \infty)$, where $C(\psi_0)$ is a positive constant depending on ψ_0 .

Condition (i) deals only with states below the ionization threshold, while (i') does not specify the dense set of ψ_0 's and, as a result, can be verified for the massless spin-boson model by modifying slightly the proof of De Roeck and Kupiainen in [2]. Hence the asymptotic completeness in this case holds with no implicit conditions.

To verify (1.6) for the spin-boson model, we proceed precisely in the same way as in [2], but using the stronger condition on the decay of correlation functions,

$$\int_0^\infty dt (1+t)^\alpha |h(t)| < \infty, \quad \text{with} \quad h(t) := \int_{\mathbb{R}^3} dk e^{-it|k|} (1 + |k|^{-1}) |g(k)|^2, \quad (1.7)$$

for some $\alpha \geq 1$, instead of Assumption A of [2], and bounding the observable $(1 + \kappa d\Gamma(\omega^{-1/2}))^2$ instead of $e^{\kappa N}$. Assumption C of [2] on initial states has to be replaced in the same manner. Assuming that (1.3) is satisfied with $\mu > 0$ (and $\eta = 1$), we see that (1.7) holds with $\alpha = 1 + 2\mu$.

The form of the observable $e^{\kappa N}$ enters [2] through the estimate $\|K_{u,v}\|_\diamond \leq \lambda^2 C |h(u-v)|$ of the operator $K_{u,v}$ defined in [2, (3.4)] and the standard estimate [2, (4.36)]. Both extend readily to our case (the former, with $h(t)$ given in (1.7)). Moreover, [2, (4.36)] is used in the proof that pressure vanishes - Eq (4.39) in [2] - and the latter also follows from our Proposition A.1. (We can also use the observable $\Gamma(\omega^{-\lambda}) = d\Gamma(-\lambda \ln \omega)$ and analyticity - rather than perturbation - in λ).

Now we comment on the modifications needed in order to prove the result of Theorem 1.3 of [3] under the new assumptions. These modifications concern only the proof of the existence of the Deift-Simon wave operators given in Theorem 5.1 of [3].

- To prove Theorem 5.1 under Assumption (i), we need minor modifications in the proof, relying on slightly strengthened Lemma 5.2, by using a new estimate on the growth of the observable N^2 (in addition to N).
- The proof of Theorem 5.1 under Assumption (i') is analogous to the one for Assumption (i). The only difference is that we do not need to introduce an artificial cutoff in the number operator. Instead we use additional 'weighted' propagation estimates, which are straightforward modifications of the estimates (3.3)–(3.4) in [3].

In the next two sections we present detailed modifications in our proof in [3], needed to prove asymptotic completeness for Rayleigh scattering under either Assumption (i) or (i').

We use the notation $\|\psi\|_\rho^2 := \|(d\Gamma(\omega^\rho) + 1)^{\frac{1}{2}}\psi_0\|$ from [3].

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2. ADJUSTMENTS IN PROOF OF THEOREM 5.1 UNDER CONDITION (i)

The part of Theorem 5.1 which requires a modification is showing that

- the family $W(t) := e^{i\hat{H}t}\check{\Gamma}(j)e^{-iHt}$ form a strong Cauchy sequence as $t \rightarrow \infty$.

We present here the corresponding changes. Let $\psi_0 \in f(H)D(d\Gamma(\omega^{-1})^{1/2})$, $f \in C_0^\infty((E_{\text{gs}}, \Sigma))$. Lemma 2.1, proven below, implies that

$$W(t)\psi_0 = e^{i\hat{H}t}f_1(\hat{H})\check{\Gamma}(j)e^{-iHt}f_1(H)\psi_0 + \mathcal{O}(t^{-\alpha+\frac{1}{2+\mu}}\|\psi_0\|_{-1}), \quad (2.1)$$

where $f_1 \in C_0^\infty((E_{\text{gs}}, \Sigma))$ is such that $f_1 f = f$. Hence, since our conditions on α imply $\alpha > 1/(2+\mu)$, it suffices to show that

$$\widetilde{W}(t) := e^{i\hat{H}t}f_1(\hat{H})\check{\Gamma}(j)e^{-iHt}f_1(H)$$

form a strong Cauchy sequence as $t \rightarrow \infty$. This is done exactly as in [3] for $W(t)$. It remains to prove the following lemma which is strengthening of the corresponding lemma (Lemma 5.2) of [3].

The rest of the proof of Theorem 5.1 of [3] under Assumption (i) is exactly the same as in [3]. \square

Lemma 2.1. *Assume (1.3) with $\mu > 0$ and (1.4). For any $f \in C_0^\infty(\Delta)$, $\Delta \subset (E_{\text{gs}}, \Sigma)$, and $\psi_0 \in \text{Ran } E_\Delta(H) \cap D(d\Gamma(\omega^{-1})^{1/2})$,*

$$\|(\check{\Gamma}(j)f(H) - f(\hat{H})\check{\Gamma}(j))\psi_t\| \lesssim t^{-\alpha+\frac{1}{2+\mu}}\|\psi_0\|_{-1}. \quad (2.2)$$

Proof. Using the Helffer-Sjöstrand formula, we compute $\check{\Gamma}(j)f(H)\psi_t - f(\hat{H})\check{\Gamma}(j)\psi_t = R$, where

$$R := \frac{1}{\pi} \int \partial_{\bar{z}} \tilde{f}(z)(\hat{H} - z)^{-1}(\hat{H}\check{\Gamma}(j) - \check{\Gamma}(j)H)(H - z)^{-1}\psi_t \, d\text{Re } z \, d\text{Im } z, \quad (2.3)$$

and \tilde{f} is an almost analytic extension of f with the usual properties. We have $\hat{H}\check{\Gamma}(j) - \check{\Gamma}(j)H = \tilde{G}_0 - iG_1$, where $\tilde{G}_0 := U d\Gamma(j, \underline{\omega}j - j\omega)$ and $G_1 := (I(g) \otimes \mathbf{1})\check{\Gamma}(j) - \check{\Gamma}(j)I(g)$.

We consider \tilde{G}_0 . We have $\underline{\omega}j - j\omega = ([\omega, j_0], [\omega, j_\infty])$, and, by Corollary B.3 of Appendix B of [3],

$$[\omega, j_\#] = \frac{\theta_\epsilon}{ct^\alpha} j'_\# + r, \quad (2.4)$$

where $j_\#$ stands for j_0 or j_∞ , $j'_\#$ is the derivative of $j_\#$ as a function of $\frac{b_\epsilon}{ct^\alpha}$, and r satisfies $\|r\| \lesssim t^{-2\alpha+\kappa}$. Since $\theta_\epsilon \leq 1$ and since $\kappa < \alpha$, we deduce that $[\omega, j_\#] = \mathcal{O}(t^{-\alpha})$. By (C.2) of Appendix C of [3], we then obtain that

$$\|\tilde{G}_0(N+1)^{-1}\| = \|(\hat{N}+1)^{-\frac{1}{2}}\tilde{G}_0(N+1)^{-\frac{1}{2}}\| \lesssim t^{-\alpha}.$$

The equality above follows from $(\hat{N}+1)^{-1/2}\tilde{G}_0 = \tilde{G}_0(N+1)^{-1/2}$. Using, for instance, that $H \in C^1(N)$, we verify that $\|(N+1)(H-z)^{-1}(N+1)^{-1}\| \lesssim |\text{Im } z|^{-2}$, and hence

$$\|\tilde{G}_0(H-z)^{-1}\psi_t\| \lesssim t^{-\alpha}|\text{Im } z|^{-2}\|(N+1)\psi_t\|. \quad (2.5)$$

Now, we need the following result, which is a consequence of the low-momentum bound (A.1) of [3] and whose proof is given below: Under (1.3) with $\mu > 0$, we have that

$$\|N\psi_t\| \lesssim t^{\frac{1}{2+\mu}}\|\psi_0\|_{-1}, \quad (2.6)$$

provided $\psi_0 \in f(H)D(d\Gamma(\omega^{-1})^{1/2})$, with $f \in C_0^\infty(\mathbb{R})$. Applying this estimate, we obtain

$$\|\tilde{G}_0(H-z)^{-1}\psi_t\| \lesssim t^{-\alpha+\frac{1}{2+\mu}}|\text{Im } z|^{-2}\|\psi_0\|_{-1}. \quad (2.7)$$

As in (5.30)–(5.31) of [3], we have in addition

$$\|G_1(N+1)^{-\frac{1}{2}}E_\Delta(H)\| \lesssim t^{-(\mu+\frac{3}{2})\alpha},$$

and hence, using, as above, that $\|(N+1)^{1/2}(H-z)^{-1}(N+1)^{-1/2}\| \lesssim |\operatorname{Im} z|^{-2}$, we obtain

$$\|G_1(H-z)^{-1}\psi_t\| \lesssim t^{-(\mu+\frac{3}{2})\alpha} |\operatorname{Im} z|^{-2} \|\psi_0\|_N. \quad (2.8)$$

From (2.3), (2.7), (2.8), the properties of the almost analytic extension \tilde{f} and the estimate $\|(H-z)^{-1}\| \lesssim |\operatorname{Im} z|^{-1}$, we conclude that (2.2) holds

Finally we prove (2.6). By the Cauchy-Schwarz inequality, we have $N^2 \leq d\Gamma(\omega)d\Gamma(\omega^{-1})$, and hence

$$\begin{aligned} \langle N^2 \rangle_{\psi_t} &\leq \langle d\Gamma(\omega^{-1})^{\frac{1}{2}} d\Gamma(\omega) d\Gamma(\omega^{-1})^{\frac{1}{2}} \rangle_{\psi_t} \\ &= \langle d\Gamma(\omega^{-1})^{\frac{1}{2}} d\Gamma(\omega) (H - E_{\text{gs}} + 1)^{-1} d\Gamma(\omega^{-1})^{\frac{1}{2}} (H - E_{\text{gs}} + 1) \rangle_{\psi_t} \\ &\quad + \langle d\Gamma(\omega^{-1})^{\frac{1}{2}} d\Gamma(\omega) [d\Gamma(\omega^{-1})^{\frac{1}{2}}, (H - E_{\text{gs}} + 1)^{-1}] (H - E_{\text{gs}} + 1) \rangle_{\psi_t}. \end{aligned}$$

Under Assumption (1.3) with $\mu > 0$, one verifies that $d\Gamma(\omega)[d\Gamma(\omega^{-1})^{\frac{1}{2}}, (H - E_{\text{gs}} + 1)^{-1}]$ is bounded. Since $d\Gamma(\omega)(H - E_{\text{gs}} + 1)^{-1}$ is also bounded, we obtain

$$\begin{aligned} \langle N^2 \rangle_{\psi_t} &\lesssim \|d\Gamma(\omega^{-1})^{\frac{1}{2}} \psi_t\| (\|d\Gamma(\omega^{-1})^{\frac{1}{2}} (H - E_{\text{gs}} + 1) \psi_t\| \\ &\quad + \|(H - E_{\text{gs}} + 1) \psi_t\|). \end{aligned} \quad (2.9)$$

Applying Proposition A.1 of [3] gives

$$\|d\Gamma(\omega^{-1})^{\frac{1}{2}} \psi_t\| \lesssim t^{\frac{1}{2+\mu}} \|\psi_0\| + \|d\Gamma(\omega^{-1})^{\frac{1}{2}} \psi_0\|, \quad (2.10)$$

and

$$\begin{aligned} \|d\Gamma(\omega^{-1})^{\frac{1}{2}} (H - E_{\text{gs}} + 1) \psi_t\| &\lesssim t^{\frac{1}{2+\mu}} \|\psi_0\| + \|d\Gamma(\omega^{-1})^{\frac{1}{2}} (H - E_{\text{gs}} + 1) \psi_0\| \\ &\lesssim t^{\frac{1}{2+\mu}} \|\psi_0\| + \|d\Gamma(\omega^{-1})^{\frac{1}{2}} \psi_0\|, \end{aligned} \quad (2.11)$$

where we used in the last inequality that $d\Gamma(\omega^{-1})^{\frac{1}{2}} \tilde{f}(H) d\Gamma(\omega^{-1})^{-\frac{1}{2}}$ is bounded for any $\tilde{f} \in C_0^\infty(\mathbb{R})$ (this can be verified, for instance, by using that $H \in C^1(d\Gamma(\omega^{-1}))$). Combining (2.9), (2.10) and (2.11), we obtain (2.6). This completes the proof of Lemma 2.1. \square

3. THE PROOF OF THE EXISTENCE OF W_+ UNDER ASSUMPTION (i')

The proof of the existence of W_+ under Assumption (i') is similar to the proof under Assumption (i), except that we do not need to introduce the cutoff χ_m . We use instead the following weighted propagation estimates, which are straightforward extensions of the estimates of Theorem 3.1 of [3]:

$$\int_1^\infty dt \, t^{-\beta} \|d\Gamma(\rho_1^* \chi_{\frac{b_\epsilon}{ct^\beta}=1} \rho_1)^{\frac{1}{2}} \psi_t\|^2 \lesssim \|\psi_0\|^2, \quad (3.1)$$

for μ and β as in Theorem 3.1 and any $\psi_0 \in \mathcal{H}$, and, if in addition Assumption (i') holds,

$$\int_1^\infty dt \, t^{-\beta} \|d\Gamma(\omega^{-1/2} \chi_{\frac{b_\epsilon}{ct^\beta}=1} \omega^{-1/2})^{\frac{1}{2}} \psi_t\|^2 \lesssim C(\psi_0), \quad (3.2)$$

and

$$\int_1^\infty dt \, t^{-\beta} \|d\Gamma(\rho_{-1}^* \chi_{\frac{b_\epsilon}{ct^\beta}=1} \rho_{-1})^{\frac{1}{2}} \psi_t\|^2 \lesssim C(\psi_0), \quad (3.3)$$

for any $\psi_0 \in \mathcal{D}$. Here $\rho_\nu := \chi \theta_\epsilon^{1/2} \omega^{\nu/2}$ (recall that $\chi \equiv \chi_{(\frac{|y|}{\epsilon t})^2 \leq 1}$). Likewise, under Assumption (i') the proof of the maximal velocity estimate of [1], in the form (1.9) of [3], can easily be extended to the following weighted maximal velocity estimate:

$$\|d\Gamma(\omega^{-1/2} \chi_{|y| \geq \bar{c}t} \omega^{-1/2})^{\frac{1}{2}} \psi_t\| \lesssim t^{-\gamma} (\|(d\Gamma(\omega^{-1/2} \langle y \rangle \omega^{-1/2}) + 1)^{\frac{1}{2}} \psi_0\| + C(\psi_0)), \quad (3.4)$$

for any $\bar{c} > 1$, $\gamma < \min(\frac{\mu}{2} \frac{\bar{c}-1}{2\bar{c}-1}, \frac{1}{2})$ and $\psi_0 \in \mathcal{D} \cap D(d\Gamma(\omega^{-1/2}\langle y \rangle \omega^{-1/2})^{\frac{1}{2}})$.

We only mention that to obtain for instance (3.2), we estimate the interaction term using the estimate (2.11) of [3] with $\delta = -1/2$ together with Lemma B.6 of Appendix B of [3] and (1.6).

Now, let $\psi_0 \in \mathcal{D} \cap D(d\Gamma(\omega^{-1/2}\langle y \rangle \omega^{-1/2})^{\frac{1}{2}})$. We decompose $(\widetilde{W}(t') - \widetilde{W}(t))\psi_0$ as in Equations (5.15)–(5.20) of [3]. Using the commutator estimates of Appendix B of [3] and Hardy's inequality, we verify that

$$\rho_{-1}^*(j'_0, j'_\infty)\rho_1 = \theta_\epsilon^{1/2}\chi(j'_0, j'_\infty)\chi\theta_\epsilon^{1/2} + \mathcal{O}(t^{-\alpha+(1+\kappa)/2}),$$

and likewise for the remainder terms rem_t . Hence Equations (5.19)–(5.20) of [3] can be transformed into

$$\underline{dj} = \frac{1}{ct^\alpha}\rho_1^*(j'_0, j'_\infty)\rho_{-1} + \omega^{1/2}\text{rem}'_t\omega^{-1/2} \quad (3.5)$$

$$\text{rem}'_t = \text{rem}_t + \mathcal{O}(t^{-2\alpha+(1+\kappa)/2}), \quad (3.6)$$

where rem_t is given in (5.20) of [3]. These relations give

$$G_0 = \widetilde{G}'_0 + \text{Rem}'_t, \quad (3.7)$$

where $\widetilde{G}'_0 := \frac{1}{ct^\alpha}Ud\Gamma(j, \widetilde{\mathcal{E}}_t)$, with $\widetilde{\mathcal{E}}_t = (\widetilde{c}_0, \widetilde{c}_\infty) := (\rho_1^*j'_0\rho_{-1}, \rho_1^*j'_\infty\rho_{-1})$, and

$$\text{Rem}'_t := G_0 - \widetilde{G}'_0 = Ud\Gamma(j, \text{rem}'_t).$$

Next, we consider $\widetilde{A} = \sup_{\|\hat{\phi}_0\|=1} |\int_t^{t'} ds \langle \hat{\phi}_s, G_0 \psi_s \rangle|$, where $\hat{\phi}_s = e^{-i\hat{H}s} f(\hat{H}) \hat{\phi}_0$. Let

$$\begin{aligned} a_0 &= \rho_1^*|j'_0|^{1/2}, & b_0 &= |j'_0|^{1/2}\rho_{-1}, \\ a_\infty &= \rho_1^*|j'_\infty|^{1/2}, & b_\infty &= |j'_\infty|^{1/2}\rho_{-1}. \end{aligned}$$

We have $\widetilde{c}_0 = -a_0b_0$, $\widetilde{c}_\infty = a_\infty b_\infty$. Exactly as for (C.1) of Appendix C of [3], one can show that, if $c = (a_0b_0, a_\infty b_\infty)$, where $a_0, b_0, a_\infty, b_\infty$ are operators on \mathfrak{h} , then

$$\begin{aligned} |\langle \hat{\phi}, d\widetilde{\Gamma}(j, c)\psi \rangle| &\leq \|d\Gamma(a_0a_0^*)^{\frac{1}{2}} \otimes \mathbf{1}\hat{\phi}\| \|d\Gamma(b_0^*b_0)^{\frac{1}{2}}\psi\| \\ &\quad + \|\mathbf{1} \otimes d\Gamma(a_\infty a_\infty^*)^{\frac{1}{2}}\hat{\phi}\| \|d\Gamma(b_\infty^*b_\infty)^{\frac{1}{2}}\psi\|. \end{aligned} \quad (3.8)$$

Hence \widetilde{G}'_0 satisfies

$$\begin{aligned} |\langle \hat{\phi}, \widetilde{G}'_0 \psi \rangle| &\leq \frac{1}{ct^\alpha} (\|d\Gamma(a_0a_0^*)^{\frac{1}{2}} \otimes \mathbf{1}\hat{\phi}\| \|d\Gamma(b_0^*b_0)^{\frac{1}{2}}\psi\| \\ &\quad + \|\mathbf{1} \otimes d\Gamma(a_\infty a_\infty^*)^{\frac{1}{2}}\hat{\phi}\| \|d\Gamma(b_\infty^*b_\infty)^{\frac{1}{2}}\psi\|). \end{aligned} \quad (3.9)$$

By the Cauchy-Schwarz inequality, (3.9) implies

$$\begin{aligned} \int_t^{t'} ds |\langle \hat{\phi}_s, \widetilde{G}'_0 \psi_s \rangle| &\lesssim \left(\int_t^{t'} ds s^{-\alpha} \|d\Gamma(a_0a_0^*)^{\frac{1}{2}} \otimes \mathbf{1}\hat{\phi}_s\|^2 \right)^{\frac{1}{2}} \left(\int_t^{t'} ds s^{-\alpha} \|d\Gamma(b_0^*b_0)^{\frac{1}{2}}\psi_s\|^2 \right)^{\frac{1}{2}} \\ &\quad + \left(\int_t^{t'} ds s^{-\alpha} \|\mathbf{1} \otimes d\Gamma(a_\infty a_\infty^*)^{\frac{1}{2}}\hat{\phi}_s\|^2 \right)^{\frac{1}{2}} \left(\int_t^{t'} ds s^{-\alpha} \|d\Gamma(b_\infty^*b_\infty)^{\frac{1}{2}}\psi_s\|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Since $a_0a_0^*$ and $a_\infty a_\infty^*$ are of the form $\rho_1^*\chi_{b_\epsilon=ct^\alpha}\rho_1$, the weighted minimal velocity estimate (3.3) implies

$$\int_1^\infty ds s^{-\alpha} \|\widehat{d\Gamma}(c_{\#1}c_{\#1}^*)^{\frac{1}{2}}\hat{\phi}_s\|^2 \lesssim \|\hat{\phi}_0\|^2,$$

where $\widehat{d\Gamma}(c_{\#1}c_{\#1}^*)^{\frac{1}{2}}$ stands for $d\Gamma(a_0a_0^*)^{\frac{1}{2}} \otimes \mathbf{1}$ or $\mathbf{1} \otimes d\Gamma(a_\infty a_\infty^*)^{\frac{1}{2}}$. Likewise, since $b_0^*b_0$ and $b_\infty^*b_\infty$ are of the form $\rho_{-1}^*\chi_{b_\epsilon=ct^\alpha}\rho_{-1}$, the weighted minimal velocity estimate (3.1) implies

$$\int_1^\infty ds s^{-\alpha} \|d\Gamma(c_{\#2}^*c_{\#2})^{\frac{1}{2}}\psi_s\|^2 \lesssim C(\psi_0),$$

with $c_{\#2} = b_0$ or b_∞ . The last three relations give

$$\sup_{\|\hat{\phi}_0\|=1} \left| \int_t^{t'} ds \langle \hat{\phi}_s, \tilde{G}'_0 \psi_s \rangle \right| \rightarrow 0, \quad t, t' \rightarrow \infty. \quad (3.10)$$

Applying likewise Lemma C.2 of Appendix C of [3], one verifies that Rem'_t satisfies

$$\begin{aligned} |\langle \hat{\phi}, \text{Rem}'_t \psi \rangle| &\lesssim \|\hat{\phi}\| \left(t^{-2\alpha+(1+\kappa)/2} \|\text{d}\Gamma(\omega^{-1})^{\frac{1}{2}} \psi\| + t^{-1} \|\text{d}\Gamma(\omega^{-1/2} \chi j'_\infty \chi \omega^{-1/2})^{\frac{1}{2}} \psi\| \right. \\ &\quad \left. + t^{-\alpha} \|\text{d}\Gamma(\omega^{-1/2} \chi_{(\frac{|y|}{ct})^2 \geq 1}^2 \omega^{-1/2})^{\frac{1}{2}} \psi\| \right). \end{aligned}$$

Using (1.6), the weighted minimal velocity estimate (3.2) and the weighted maximal velocity estimate (3.4), we conclude that

$$\sup_{\|\hat{\phi}_0\|=1} \left| \int_t^{t'} ds \langle \hat{\phi}_s, \text{Rem}'_s \psi_s \rangle \right| \rightarrow 0, \quad t, t' \rightarrow \infty. \quad (3.11)$$

Equations (3.10) and (3.11) then imply

$$\tilde{A} = \left\| \int_t^{t'} ds f(\hat{H}) e^{i\hat{H}s} G_0 \psi_s \right\| \rightarrow 0, \quad t, t' \rightarrow \infty. \quad (3.12)$$

The estimate of G_1 is the same as in the proof of Theorem 5.1 of [3], which shows that $\widetilde{W}(t)$, and hence $W(t)$, are strong Cauchy sequences. Thus the limit W_+ exists. \square

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(J. Faupin) INSTITUT DE MATHÉMATIQUES DE BORDEAUX, UMR-CNRS 5251, UNIVERSITÉ DE BORDEAUX 1, 33405 TALENCE CEDEX, FRANCE

E-mail address: jeremy.faupin@math.u-bordeaux1.fr

(I. M. Sigal) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ON M5S 2E4, CANADA

E-mail address: im.sigal@utoronto.ca